image in this case is an anchor ring, the capacity of which is known.[*] We can also write down at once an approximate value of the capacity between a tetrahedral or a cubical conducting sheet and a small sphere at its centre. In some cases, also, we can assume without appreciable error, that the capacity[†] of the image can be computed very approximately by the formula $(S/4\pi)^{\frac{1}{2}}$ where S is the surface of the image.

The Lateral Vibrations of Sharply-pointed Bars.

By J. W. Nicholson, F.R.S., Professor of Mathematics in the University of London.

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In a preceding paper,[‡] a discussion was given of the lateral vibrations of bars of circular cross-section formed by the revolution of the curve

$$y = Ax^n$$

—when n is between the values zero and unity—about the axis of x. The matter arose in connection with the siliceous deposits found upon a certain type of sponge spicule, as discussed in a joint paper by Prof. Dendy and the present author.[§]

It is of some interest to obtain a more extended knowledge of the vibrations of solids belonging to this class, with a view to further applications. The phenomena presented change in a curious manner with the value of n, and, in certain respects, could not be foreseen in an elementary way. A discussion of the subject, in numerical terms, for an exponent n between 1 and 2 is very laborious, and in the present paper we confine attention to the case $n = 2$. This is a limiting case, which presents very exceptional features, and gives rise to a period equation of an unusual type. It illustrates clearly, at the same time, the effect of sharpening the ends of the rod beyond the point at which they are conical ($n = 1$). The rod is a free-free bar, symmetrical about its axis, and each half is obtained by the revolution of a portion of a parabola about the tangent at its vertex.

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[*] F. W. Dyson, 'Phil. Trans.,' vol. 184, p. 43.

[†] A. Russell, 'Journ. Inst. of Elec. Engin.,' vol. 55, p. 12.

[‡] 'Roy. Soc. Proc.,' A, vol. 93, p. 506 (1917).

[§] 'Roy. Soc. Proc.,' B, vol. 89, p. 573 (1917).

When rotational inertia is neglected, the differential equation regulating the vibrations is

$$\mathrm{E}\frac{\partial^2}{\partial x^2}\left(\omega \mathrm{K}^2\frac{\partial^2 y}{\partial x^2}\right)+\omega\rho\frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the displacement at distance x from the end, E, and ρ are the modulus and density, ω is the sectional area, and K its radius of gyration about the appropriate axis. For a vibration of period $2\pi/p$,

$$\partial^2 y/\partial t^2 = -p^2 y,$$

and if the generating curve is $\eta = \mathrm{A}\xi^2$ we have

$$\omega = \pi \mathrm{A}^2 x^4, \qquad \mathrm{K}^2 = \tfrac{1}{4}\mathrm{A}^2 x^4,$$

whence
$$\frac{\partial^2}{\partial x^2}\left(x^8\frac{\partial^2 y}{\partial x^2}\right) = qx^4 y, \tag{1}$$

where
$$q = 4\rho^2/\mathrm{EA}^2. \tag{2}$$

If we use the operation $\vartheta \equiv x\dfrac{\partial}{\partial x}$ the equation becomes

$$\vartheta(\vartheta-1)(\vartheta+5)(\vartheta+6)y = qy$$

and the solution is

$$y = \mathrm{A}_1 x^{\alpha_1} + \mathrm{A}_2 x^{\alpha_2} + \mathrm{A}_3 x^{\alpha_3} + \mathrm{A}_4 x^{\alpha_4}$$

where the indices α are the roots of the biquadratic in m,

$$m(m-1)(m+5)(m+6) = q. \tag{3}$$

This case is, in fact, the only one in which the value of y is compounded of simple functions. The subsequent analysis of the periods recalls, in some respects, that of a similar problem of variable strings discussed by Lord Rayleigh, in that functions of the type $\cos(\beta \log x)$ are predominant.

Evidently, no vibrations are possible unless the equation (3) has two or four non-real roots, which can introduce trigonometric functions of $\log x$ into the value of y.

The quartic can be written in the form

$$(m^2+5m)(m^2+5m-6) = q,$$

whence
$$m^2+5m = 3 \pm \surd\{q+9\}. \tag{4}$$

If
$$q+9 = (3+t)^2 \tag{5}$$

where t is essentially positive, the values of m ultimately become

$$m = -\tfrac{5}{2} \pm \tfrac{1}{2}\surd\{49+4t\}, \qquad m = -\tfrac{5}{2} \pm \tfrac{1}{2}\surd\{25-4t\}. \tag{6}$$

The first pair are necessarily real. The second pair cannot be complex unless t exceed $6\tfrac{1}{4}$, or q is greater than 76·5625. This gives a lower limit to the frequency of which the bar is capable under any conditions of support. This limiting case gives

$$\frac{p}{2\pi} = \frac{4{\cdot}375\,\mathrm{A}}{2\pi}\sqrt{\frac{\mathrm{E}}{\rho}}. \tag{7}$$

This frequency is seen later to be associated with a mathematically sharp end.

If this condition be satisfied, and if

$$(\alpha_1, \alpha_2) = -\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{\{49+4t\}}, \tag{8}$$

$$\gamma^2 = t - \tfrac{25}{4}, \tag{9}$$

then the value of y is

$$y = \mathrm{A}_1 x^{\alpha_1} + \mathrm{A}_2 x^{\alpha_2} + x^{-5/2}\{\mathrm{A}_3 \cos(\gamma \log x) + \mathrm{A}_4 \sin(\gamma \log x)\}. \tag{10}$$

Let the bar have a free end at $x = \epsilon$, where ϵ may be made small at a later stage. The shearing stress and bending moment at this point must vanish, so that, when $x = \epsilon$,

$$x^8 \frac{\partial^2 y}{\partial x^2} = 0, \qquad \frac{\partial}{\partial x}\left(x^8 \frac{\partial^2 y}{\partial x^2}\right) = 0. \tag{11}$$

We shall, as stated, illustrate the analysis only by a free-free bar, composed of two halves turned opposite ways, and joined at the greatest cross-section. Each is obtained by rotating the curve $y = \mathrm{A}x^2$, between $x = \epsilon$ and $x = l$, round the axis of x. The length of the bar is $2(l-\epsilon)$. We also limit attention to the symmetrical vibrations, in which the two ends are moving alike at any instant.

In such vibrations there is no shearing stress at $x = l$, and, moreover, dy/dx is zero by symmetry. These conditions, with (11), determine the period equation.

If $y = y_1 + y_2$, where

$$y_1 = x^{-5/2}\{\mathrm{A}_3 \cos(\gamma \log x) + \mathrm{A}_4 \sin(\gamma \log x)\}, \tag{12}$$

$$y_2 = \mathrm{A}_1 x^{\alpha_1} + \mathrm{A}_2 x^{\alpha_2}, \tag{13}$$

we find, with the use of the values of α, β, γ above that

$$\frac{\partial^2 y_1}{\partial x^2} = -\frac{6}{x}\frac{\partial y_1}{\partial x} - \left(\gamma^2 + \tfrac{25}{4}\right)\frac{y_1}{x^2},$$

$$\frac{\partial}{\partial x}\left(x^8 \frac{\partial^2 y_1}{\partial x^2}\right) = -x^6\left(\gamma^2 + \tfrac{49}{4}\right)\frac{\partial y_1}{\partial x},$$

$$\frac{\partial^2 y_2}{\partial x^2} = -\frac{6}{x}\frac{\partial y_2}{\partial x} + (6+t)\frac{y_2}{x^2} = -\frac{6}{x}\frac{\partial y_2}{\partial x} + \left(\gamma^2 + \tfrac{49}{4}\right)\frac{y_2}{x^2},$$

$$\frac{\partial}{\partial x}\left(x^8 \frac{\partial^2 y_2}{\partial x^2}\right) = t x^6 \frac{\partial y_2}{\partial x} = x^6\left(\gamma^2 + \tfrac{25}{4}\right)\frac{\partial y_2}{\partial x}.$$

Thus the conditions at the centre of the bar become

$$\frac{\partial y_1}{\partial x}+\frac{\partial y_2}{\partial x}=0,$$

$$(\gamma^2+\tfrac{49}{4})\frac{\partial y_1}{\partial x}-(\gamma^2+\tfrac{25}{4})\frac{\partial y_2}{\partial x}=0,$$

or, more briefly, $\partial y_1/\partial x$ and $\partial y_2/\partial x$ separately vanish when $x=l$. This is a considerable gain in simplicity. These conditions are satisfied by the form

$$y=\mathrm{H}\left\{\frac{x^{\alpha_1}}{\alpha_1 l^{\alpha_1}}-\frac{x^{\alpha_2}}{\alpha_2 l^{\alpha_2}}\right\}+\mathrm{K}x^{-5/2}\left\{2\gamma\cos\left(\gamma\log\frac{x}{l}\right)+5\sin\left(\gamma\log\frac{x}{l}\right)\right\}, \quad (14)$$

where H and K are arbitrary. This expression applies to any bar of this type vibrating symmetrically about its centre, whatever the conditions at the ends. For a free end at $x=\epsilon$ and at $x=2l-2\epsilon$, the further conditions (11) become

$$(\gamma^2+\tfrac{25}{4})\frac{\partial y_2}{\partial x}=(\gamma^2+\tfrac{49}{4})\frac{\partial y_1}{\partial x},$$

$$6x\left(\frac{\partial y_1}{\partial x}+\frac{\partial y_2}{\partial x}\right)=(\gamma_2+\tfrac{49}{4})y^2-(\gamma^2+\tfrac{25}{4})y_1,$$

at $x=\epsilon$, where

$$x\frac{\partial y_2}{\partial x}=\mathrm{H}\left\{\left(\frac{\epsilon}{l}\right)^{\alpha_1}-\left(\frac{\epsilon}{l}\right)^{\alpha_2}\right\},$$

$$x\frac{\partial y_1}{\partial x}=-2(\gamma^2+\tfrac{25}{4})\mathrm{K}\sin\left(\gamma\log\frac{\epsilon}{l}\right).\ \epsilon^{-5/2}$$

on reduction. Thus using (18) below, by which

$$(\gamma^2+49)/(\alpha_1,\alpha_2)=5+\alpha_1,\ 5+\alpha_2,$$

we find ultimately

$$\mathrm{H}\left\{\left(\frac{\epsilon}{l}\right)^{\alpha_1}-\left(\frac{\epsilon}{l}\right)^{\alpha_2}\right\}=-2(\gamma^2+\tfrac{49}{4})\mathrm{K}\sin\left(\gamma\log\frac{\epsilon}{l}\right).\ \epsilon^{-5/2}, \quad (15)$$

$$\mathrm{H}\left\{(\alpha_1-1)\left(\frac{\epsilon}{l}\right)^{\alpha_1}-(\alpha_2-1)\left(\frac{\epsilon}{l}\right)^{\alpha_2}\right\}$$

$$=(\gamma^2+\tfrac{25}{4})\left\{2\gamma\cos\left(\gamma\log\frac{\epsilon}{l}\right)-7\sin\left(\gamma\log\frac{\epsilon}{l}\right)\right\}\mathrm{K}\epsilon^{-5/2}, \quad (16)$$

and therefore

$$\gamma\cot\left(\gamma\log\frac{\epsilon}{l}\right)-\tfrac{7}{2}=-\left(\frac{4\gamma^2+49}{4\gamma^2+25}\right)\left\{\frac{(\alpha_2-1)-(\alpha_1-1)(\epsilon/l)^{\alpha_1-\alpha_2}}{1-(\epsilon/l)^{\alpha_1-\alpha_2}}\right\}. \quad (17)$$

This is the period equation. If the ratio ϵ/l is fixed, its solution gives the possible values of γ, and thence of the frequency, for α_1 and α_2 may be expressed in terms of γ by the equations (8, 9), which lead to

$$(\alpha_1,\alpha_2)=-\tfrac{5}{2}\pm\tfrac{1}{2}\sqrt{\{74+4\gamma^2\}}, \quad (18)$$

one being positive, and the other negative. Choose α_2 as the negative value and

$$\alpha_1-\alpha_2=\sqrt{\{74+4\gamma^2\}}, \tag{19}$$

which must be greater than 8. When ϵ is only comparatively small in relation to l, the ratio

$$\{(\alpha_2-1)-(\alpha_1-1)(\epsilon/l)^{\alpha_1-\alpha_2}\}/\{1-(\epsilon/l)^{\alpha_1-\alpha_2}\}$$

may be taken as α_2-1. For example, if the sectional area at one end of the rod is as much as one-quarter of the central sectional area, an error is involved only of magnitude 1 in 2^8 in the most unfavourable case. Thus the period equation may be reduced to

$$2\gamma\cot\left(\gamma\log\frac{\epsilon}{l}\right)-7=\left(\frac{4\gamma^2+49}{4\gamma^2+25}\right)\{7+\sqrt{(74+4\gamma^2)}\}. \tag{20}$$

Returning to (5) and (9), we notice that

$$q=(3+t)^2-9=(\gamma^2+\tfrac{37}{4})^2-9$$

and

$$q=4\rho p^2/\mathrm{EA}^2,$$

where, if r_1 and r_2 are the radii of section at the end and at the centre,

$$r_1=\mathrm{A}\epsilon^2,\qquad r_2=\mathrm{A}l^2,\qquad \mathrm{A}=(\sqrt{r_2}-\sqrt{r_1})^2/(l-\epsilon)^2, \tag{21}$$

where $l-\epsilon$ is half the length of the rod. The frequency corresponding to any root γ of (20) is

$$\frac{p}{2\pi}=\frac{1}{16\pi}\sqrt{\frac{\mathrm{E}}{\rho}}\cdot\left(\frac{\sqrt{r_2}-\sqrt{r_1}}{l-\epsilon}\right)^2\sqrt{\{(4\gamma^2+25)(4\gamma^2+49)\}}, \tag{22}$$

and in the period equation (20), $\log\epsilon/l$ may be replaced by $\frac{1}{2}\log r_1/r_2$.

Periods of the Rod.—The equation for γ may be written in the form

$$-\tan\left(\gamma\log\frac{l}{\epsilon}\right)=\frac{2\gamma(4\gamma^2+25)}{518+56\gamma^2+(4\gamma^2+49)\sqrt{(4\gamma^2+74)}}, \tag{23}$$

where the expression on the right is always necessarily real and positive, lying between zero and unity. Thus, for the first root of the equation, $\gamma\log l/\epsilon$ is an angle in the second quadrant, and, for the first vibration,

$$3\pi/4<\gamma\log l/\epsilon<\pi,$$

while for the second

$$7\pi/4<\gamma\log l/\epsilon<2\pi$$

and so on, whatever be the value of the logarithm. As this logarithm increases, or the rod becomes sharper, the intervals between the frequencies of successive notes become shorter, all the values of γ tending towards zero. For an indefinitely sharp rod, with $\epsilon=0$, we reach the curious conclusion that all the notes tend to the same frequency, which is the only frequency the bar can produce. A complete change in behaviour in this respect accompanies the passage from a double conical bar to one of the present type. This case could not, of course, be realised in practice, where, for a very sharp end, only a set of nearly equal frequencies would be obtained. The tendency to equality is only logarithmic.

When the ends are only fairly sharp, all the initial values of γ are small, and effectively given by

$$-\tan(\gamma \log l/\epsilon) = 0{\cdot}0532\gamma + 0{\cdot}00275\gamma^3, \tag{24}$$

whose solution, in a rapidly convergent form, is

$$\gamma = \frac{s\pi}{\log l/\epsilon} - \frac{0{\cdot}0532\, s\pi}{(\log l/\epsilon)^2} + \frac{0{\cdot}00283\, s^2\pi^2}{(\log l/\epsilon)^3} - \frac{0{\cdot}00285\, s^3\pi^3}{(\log l/\epsilon)^4} + \ldots, \tag{25}$$

obtained readily after the usual manner, where s takes integral values, excluding zero.

The limiting frequency corresponding to mathematically sharp ends is given by $\gamma = 0$ and $r_1 = 0$. It becomes

$$\frac{p}{2\pi} = \frac{35}{16\pi} \cdot \frac{r_2}{l^2} \cdot \sqrt{\frac{\mathrm{E}}{\rho}},$$

and is in fact the limiting frequency of which the bar is capable, as discussed in an earlier part of the paper (equation 7). When the ends are only moderately sharp, the frequency of the fundamental note, at least, approximates very closely to this value. It is dependent on the value of

$$\eta = \sqrt{\{(4\gamma^2 + 25)(4\gamma^2 + 49)\}}. \tag{26}$$

The values of η for the first three notes are exhibited in the following Table (Table I) for various degrees of sharpness of the rod:—

Table I.

	First note.		Second note.		Third note.	
$\mathrm{Log}_e l/\epsilon$.	γ.	$4\eta/35$.	γ.	$4\eta/35$.	γ.	$4\eta/35$.
4	0·775	1·072	1·549	1·286	2·319	1·633
5	0·620	1·046	1·241	1·1845	1·863	1·413
6	0·519	1·032	1·039	1·1295	1·556	1·289
7	0·445	1·024	0·890	1·095	1·336	1·213
8	0·390	1·018	0·780	1·073	1·170	1·164

This Table shows the slow convergence of η to the value 35/4. The higher the note, the slower is this convergence. We may use the formula

$$\frac{p}{2\pi} = \frac{35}{16\pi} \cdot \frac{\mathrm{R}\alpha}{\mathrm{L}^2} \sqrt{\frac{\mathrm{E}}{\rho}}, \tag{27}$$

for the frequencies, where

$$\alpha = \frac{4\eta}{35}\left(1 - \sqrt{\frac{r}{\mathrm{R}}}\right)^2, \tag{28}$$

and R, L are the central radius and semi-length of the bar, while r is the radius at the ends. Table II exhibits the values of α for various small values of the ratio r/R, or, in other words, the extent to which finite thickness at the ends is significant.

Table II.

	First note.	Second Note.	Third Note.
r/R.	α.	α.	α.
e^{-8}	1·033	1·239	1·574
e^{-10}	1·032	1·169	1·394
e^{-12}	1·027	1·124	1·283
e^{-14}	1·022	1·093	1·211
e^{-16}	1·017	1·072	1·163

It is evident that although the sharpness of the ends is not very important as a contributing factor to the fundamental frequency, which differs only slightly from its value for indefinitely sharp ends, when the end radius is only 1/3000 of that at the centre, the other frequencies are affected in a quite definite way, and are in fact very sensitive to the degree of sharpness as they recede from the fundamental. The limiting frequency would be very difficult to realise even approximately.

The Nodal Positions.—From (14) and (15), the vibration curve of the bar is of the form

$$y = \mathrm{K}x^{-5/2}\left\{2\gamma\cos\left(\gamma\log\frac{x}{l}\right)+5\sin\left(\gamma\log\frac{x}{l}\right)\right\}$$

$$-2\mathrm{K}\epsilon^{-5/2}\left(\gamma^2+\tfrac{49}{4}\right)\sin\left(\gamma\log\frac{x}{l}\right)\left\{\frac{x^{\alpha_1}}{\alpha_1 l^{\alpha_1}}-\frac{x^{\alpha_2}}{\alpha_2 l^{\alpha_2}}\right\}\Big/\left\{\left(\frac{\epsilon}{l}\right)^{\alpha_1}-\left(\frac{\epsilon}{l}\right)^{\alpha_2}\right\}, \qquad (29)$$

where $$(\alpha_1, \alpha_2) = -\tfrac{5}{2}\pm\sqrt{\left(\gamma^2+\tfrac{74}{4}\right)}$$

when ϵ/l is small, $(\epsilon/l)^{\alpha_1-\alpha_2}$ is less than $(\epsilon/l)^8$, and therefore negligible to a high order. The nodes are then given by

$$\left(\frac{x}{l}\right)^{-5/2}\left\{2\gamma\cos\left(\gamma\log\frac{x}{l}\right)+5\sin\left(\gamma\log\frac{x}{l}\right)\right\}$$

$$= 2\left(\gamma^2+\tfrac{49}{4}\right)\sin\left(\gamma\log\frac{x}{l}\right)\left\{\frac{x^{\alpha_1}}{\alpha_1 l^{\alpha_1}}-\frac{x^{\alpha_2}}{\alpha_2 l^{\alpha_2}}\right\}\left(\frac{\epsilon}{l}\right)^{-5/2-\alpha_2}. \qquad (30)$$

Since x is less than l, and α_1 and α_2 are both numerically greater than unity, the expression on the right, multiplied by $(x/l)^{5/2}$, will not exceed $49(\epsilon/l)^4$, which is quite negligible even if $\log l/\epsilon$ is only about 3 or 4.

The nodal equation for a nearly sharp rod is thus reduced effectively to

$$\tan(\gamma \log x/l) = -2\gamma/5, \tag{31}$$

where γ itself is small. If γ tends to zero, the first root tends to

$$\log x/l = -\tfrac{2}{5}, \qquad x/l = e^{-2/5} = 0{\cdot}67032. \tag{32}$$

This corresponds to the limiting vibration, which thus has nodes at distances $0{\cdot}6703l$ from each end, where l is half the length.

We shall calculate the positions of the nodes for the first three notes in two other cases, in order to indicate the effect of sharpening the ends of the rod. The simple equation (31) may be used.

CASE 1.—$\operatorname{Log} l/\epsilon = 4$.

For the first three tones we have

$$\gamma = 0{\cdot}775, \quad 1{\cdot}549, \quad 2{\cdot}319,$$

$$2\gamma/5 = 0{\cdot}3100, \quad 0{\cdot}6196, \quad 0{\cdot}9276,$$

$$\log l/x = \gamma^{-1}\tan^{-1}(2\gamma/5) = 0{\cdot}3881, \quad 0{\cdot}3582, \quad 0{\cdot}3226.$$

These are the initial values in each case. More generally, $\pm s\pi/\gamma$ must be added, where s is an integer.

Thus for the *first tone*,

$$\log l/x = 0{\cdot}3881 \pm 4{\cdot}053s.$$

When $s = 1$, the positive sign gives $x < \epsilon$, lying outside the bar, and therefore an inadmissible value. The negative sign gives $x > l$, again inadmissible. There is thus, as we should expect, only one node, corresponding to

$$\log l/x = 0{\cdot}3881, \qquad x/l = e^{-0\cdot3881} = 0{\cdot}67834, \tag{33}$$

very close to the limiting node. A similar node exists in the other half of the bar.

For the *second tone*,

$$\log l/x = 0{\cdot}3582 \pm 2{\cdot}0265s.$$

There are two possible values, (a) corresponding to $s = 0$, and (b) corresponding to $s = 1$, with the positive sign in the ambiguity. These values are

$$x/l = e^{-0\cdot3582} = 0{\cdot}6989,$$

$$x/l = e^{-2\cdot3847} = 0{\cdot}0921, \tag{34}$$

where the first is near the critical value, and the second very close to the end of the rod. Two nodes would be expected in this vibration.

For the *third tone*,

$$\log l/x = 0{\cdot}3226 \pm 1{\cdot}3544s.$$

There are three possible values, as anticipated, corresponding to $s = 0, 1, 2,$

the positive sign being necessary in the two latter cases. The resulting values are

$$x/l = e^{-0\cdot3226} = 0\cdot7243,$$
$$x/l = e^{-1\cdot6770} = 0\cdot1869,$$
$$x/l = e^{-3\cdot0314} = 0\cdot0482, \qquad (35)$$

the third being still nearer to the end, and the first diverging further from the critical position.

CASE 2.—$\log l/\epsilon = 8$.

This corresponds to a much sharper rod. We find for the first three tones

$$\gamma = 0\cdot390, \quad 0\cdot780, \quad 1\cdot170,$$
$$\tan^{-1}(2\gamma/5) = 0\cdot1547, \quad 0\cdot3020, \quad 0\cdot4377,$$
$$\log l/x = 0\cdot3968 \pm 8\cdot0544s. \quad \text{(First tone.)}$$
$$= 0\cdot3877 \pm 4\cdot0277s. \quad \text{(Second tone.)}$$
$$= 0\cdot3741 \pm 2\cdot6851s. \quad \text{(Third tone.)}$$

The resulting possible nodes are, for the first tone,

$$x/l = e^{-0\cdot3968} = 0\cdot6725. \qquad (36)$$

For the second,

$$x/l = e^{-0\cdot3877} = 0\cdot6786,$$
$$x/l = e^{-4\cdot4154} = 0\cdot0121. \qquad (37)$$

For the third,

$$x/l = e^{-0\cdot3741} = 0\cdot6879,$$
$$x/l = e^{-3\cdot0592} = 0\cdot0469,$$
$$x/l = e^{-5\cdot7443} = 0\cdot0032. \qquad (38)$$

It is clear that, as the ends of the rod become sharper, all nodes, except one corresponding to each vibration, tend towards the ends of the rod. The higher the order of the note, the more rapid is this tendency. The exceptional node of each vibration tends to the limiting position, and the numerical cases discussed serve to make the process clearer. The conclusions are those which would be expected to accord with the behaviour of the frequencies.

An investigation of the unsymmetrical vibrations of the double rod of this type has also been made, but is much more complicated, and it is not thought necessary to reproduce it. We find, again, that the values of γ tend to zero when the rod becomes progressively sharper, and all the vibrations tend to have the limiting frequency. The central conditions here are $y = 0$ and $d^2y/dx^2 = 0$, for such vibrations involve a point of inflexion at the centre.

A similar problem presenting the same features is that of a single rod formed from the generating curve $y = Ax^2$, with the thick end clamped, and subject to the conditions $y = 0$, $dy/dx = 0$. But the peculiar form of the analysis in these cases has already been indicated sufficiently, together with the unusual acoustic behaviour of the systems derived from this particular generating curve.

With reference to the transition, through $y = Ax^n$, from the values $n = 1$, the limit of the preceding paper, and $n = 2$, the value in the present paper, it is necessary to point out that interpolation is not possible, and that the solution for intermediate values of n requires asymptotic formulæ, suitable for larger values of the argument, for the functions partially discussed in the first paper. These functions are of considerable interest and generality.

A New Method of Spectrophotometry in the Visible and Ultra-violet and the Absorption of Light by Silver Bromide.

By R. E. Slade, D.Sc., F.I.C., and F. C. Toy, B.Sc.

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Introduction.

A quantitative investigation of the absorption of light by silver bromide has been undertaken as a preliminary to a photochemical investigation of the action of silver bromide in the photographic dry plate.

A good summary of the advantages and disadvantages of the various methods which have been devised by different experimenters for the quantitative investigation of the absorption of light by substances is given by Ewest in a thesis entitled, "Beiträge zur quantitativen Spectralphotographie," of which an abstract is given by F. F. Renwick.* All the methods which have been used previously either depend upon Schwarzschild's law of the relation between time of exposure and the photographic effect, or a so-called neutral wedge is used which is supposed to absorb equally in all wave-lengths or is calibrated for selective absorption. The method which we have used is in some ways similar to that used by Ewest, but the apparatus required is very much simpler and a wedge of the material under examination is used instead of the neutral wedge of Ewest. In our method all that is required of the photographic plate is that the exposure of two adjacent portions of the same plate to the same light intensity of the same wave-length

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* 'Photographic Journal,' vol. 54, p. 99 (1914).